rich, even awe inspiring. For days, a problem boggled. But then someone would offer another way of looking at it, and suddenly it would make sense.

What would it look like to teach elementary school children math in the way she was learning it? Adney, who taught undergraduates, could take her only so far. Deborah needed another resource.

A few years after her classes with Adney, Deborah decided to teach a summer school section outside her usual repertoire. She'd just taken a class on research methods, and the material had struck her as potentially powerful for eight- and nine-year-olds. In particular, she wanted to teach inferential statistics, a kind of math in which students use tools like curves and intervals to draw conclusions about data. But, finding no research or curriculum on how to teach the subject to young children, she'd had to create the course from scratch. This proved more challenging than she anticipated, so she decided to recruit help. Not a coteacher—the class had only eighteen students, a perfectly manageable number for one person. What Deborah needed was another brain. Better yet, a dozen of them.

Recruiting teachers was simple; by participating, they could cross off a required professional development session. Every day that summer, before the children arrived, the group walked through the lesson Deborah had drafted, trying out problems, imagining how students might react, and discussing what Deborah might say in response. When the lesson began, the other teachers served as extra eyes and ears, studying each child and noting what they did and did not understand. At the end of each day, Deborah had the students leave their notebooks behind so the teachers could study those too. Then they all sat together and talked about what had just happened. What did everyone

think about what this or that student had said? What ideas did the class still not seem to grasp? What should Deborah do tomorrow?

In a way, this method was no different from her normal practice. At Spartan Village, she frequently pulled other teachers into her class to help her solve problems. But at Spartan Village, moments like these were merely friendly favors offered by busy colleagues. At the summer program, the group's focus was sustained, the tone serious; it was as if they were not in an elementary school, but in a laboratory. Or maybe, Deborah thought, a surgical theater.

Technically, only Deborah taught the children. But really she was the group's surrogate—a kind of "pedagogical daredevil," she decided, trying out ideas on everyone's behalf. "Whatever we decided to do," she wrote later, "I was the one who had to try to make it fly." The group, meanwhile, formed her safety net, making sure the students didn't become casualties of the experiment.

The students learned and, just as importantly, so did Deborah. Looking back, she says it is impossible to recall any one moment of epiphany. Her teaching was evolving quickly, and she hadn't yet begun to make records capturing each lesson and the discussions that followed. But similar public lessons, held years later, at the annual program known as the Elementary Math Lab, shed light on what she and those first co-conspirators must have seen that first summer in 1984.

During a lesson in July 2012, a group of observers took notes from bleacher-style seats as Deborah asked a class of rising sixth-graders to consider a rectangle. The rectangle looked like this:
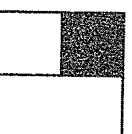
What fraction of the rectangle, Deborah asked the students, is shaded? The first student she called on, a girl named Anya, gave the correct answer, ¼, explaining how she had drawn an additional line to help her solve the problem:
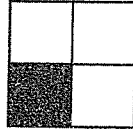
But when Deborah asked for comments on Anya's answer, a boy named Shamar, with puffy cheeks and long dreadlocks, said something curious. "I think the answer was one-half and a one on the side of it," he said. The mysteries multiplied when Deborah brought him up to the board to explain. "1 ½," he wrote. But he kept saying the number backward, as if reading from right to left: one-half first, then one, which he called the "remainder."

What was he thinking? Under what assumptions might 1½ make sense? Scrutinizing Shamar's responses during the debriefing Deborah held after the lesson, once the students had left, one group of observers pieced together a hypothesis. Perhaps he had flipped the question. Instead of looking at the fraction of the rectangle that was shaded, he focused on the fraction that was empty. He might have even seen the image as its inverse:
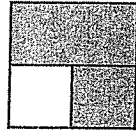
Others focused on Shamar's description of 1½ as "one-half and a one on the side of it"—more like ½ 1 than 1½. Maybe he had transcribed the inverse image into the numbers that it resembled: ½ on the left, 1 on the right. If you saw math as a set of rules and procedures, as so many children were taught, then you did not think about fractions as holding meaning. They were simply numbe... with lines through their middles.

Whatever his exact thought process was, Shamar had clearly become confused about an idea that stood at the heart of fractions—one that, over the years, Deborah and those who joined her at the lab had come to see as a typical stumbling block for children (and many adults): the idea of the whole. To answer any fractions problem, you had to define the thing that you wanted to know a fraction *of*.

In this case, the whole was the largest rectangle, the one that also happened to be a square. Shamar's answer suggested that he had defined a different rectangle as the whole—the one that, with its longer sides, looked more like a child's idea of a rectangle. (Children often do not understand that all squares are rectangles too, just with equal-length sides.) If you defined that slimmer rectangle as the whole, and you accepted Shamar's inversion of the shaded and empty space, then 1½ made sense.

The misunderstanding offered an opportunity. Encountering the math through the students' eyes, the group could figure out what needed to be clarified. Then together, these observers could figure out what Deborah might say and do to get Shamar to understand the importance of defining the whole.

There were many possible paths into the material: questions to ask, explanations to give, problems to assign. Over the course of many teaching labs, the most productive methods and problems made themselves clear. One good approach was to have students with different ideas present them to the class. Some of them undoubtedly shared Shamar's misunderstanding, in one form or another. (Even adults in the room could forget something—that a fraction was meaningless without its unit.) Listening to their peers could help the confused students sort out their ideas. When a boy named Eduardo jumped in to clarify, Shamar seemed to understand his own idea better too. Then, when Eduardo explained why he agreed with Anya anyway a[...]t ¼, Shamar decided to change his answer.

The lab group also studied turns—which students Deborah called on, in what order, and what she asked each of them to do. Her decisions had come to hold more significance over time, as she learned the many different types of turns, each with varying dimensions of both academic difficulty (offering a math fact versus offering an interpretation) and social risk (giving an answer even though you hadn't raised your hand was moderately risky; coming up to the board and offering a detailed description of your incorrect answer, much more so).

Other considerations mattered too. To make sure everyone participated, it was advisable to call on the three students who had not spoken yet, but this might not be a good strategy if all three had the same answer. Order also made a difference. In certain cases, there was wisdom to calling on, say, Anya before Shamar. Shamar's answer had assumed an idea that Anya's, by drawing in the previously invisible line, made explicit—that fractions made sense only if they formed *equal* parts of the whole. The discussion would go better if Deborah could get Anya's idea on the table before tackling Shamar's confusion.

Over time, more conventions emerged. It was crucial, for instance, to make sure that students did not talk just to Deborah, but to the entire class. Everyone had to learn everyone else's name. Then, instead of saying "that weird idea about one-half and one beside it," they could simply say "Shamar's idea" or, if Shamar posited an argument, "Shamar's *conjecture*." Deborah came to see these named conjectures as "fence posts" for a productive conversation. The students could peer backward over the landscape of their evolving understanding and name the key turning points. And when another part-whole misunderstanding inevitably arose, they could undo it more quickly by thinking back to Shamar's idea and the reasons it didn't hold up. The precise wording of questions also mattered, and the lab

group spent hours debating Deborah's constructions. That same year, hoping to introduce students to the concept of infinity—one of those dazzling ideas that could spin in a student's mind for days—she had presented a problem with endless answers. Then, asking the students to guess how many solutions they could come up with, she added an extra question, apparently as an afterthought. "After you write down your answer," she asked, "can you write how long it will take [to come up with all the solutions]?"

The lab group devoted several minutes to considering the value of that extra question. By asking the students to write down how long writing the solutions would take, hadn't Deborah suggested that writing them all down was actually possible? And so, argued a teacher from Chicago, hadn't the question inadvertently tilted the students away from the correct answer? But the question had done exactly the opposite, another group of teachers argued. "We could be doing this *forever!*" the students might realize, thereby jumping closer to the key idea.

The group dissected the problems Deborah selected too. On the day of Shamar's misunderstanding, another confusion had arisen—this one not about the whole, but about the parts. Counting the shaded part and then counting the total number of parts, some students had called the fraction ⅓. They had missed what Anya saw about drawing a line to make the parts equal. The class discussed why dividing a shape into equal parts was important, but some lab observers wondered whether all the students really grasped this idea. One person offered a proposal. In the next class, why not present a problem that forced the students to draw even more lines? Something like this:

The next day, Deborah added the problem to the warm-up.

Back at Spartan Village, the lessons from the early summer labs—which began in 1984 and continued for years after—were combining with the new curriculum to create a kind of magic. Now that the students conjectured, reasoned, argued, and proved, they were building one idea on top of another. They sometimes forgot what they'd learned, like all students do. But now, when they stumbled, they could pick themselves up. Deborah saw it happen one day a few weeks into the fractions unit, when two third-graders were puzzling over a problem about cookies that involved the number ⁴⁄₂.

"How can we have this?" Betsy asked Jeannie, pointing to the confusing fraction.

"I don't know," Jeannie said.

"Four *twoths*?" Betsy asked.

"We take something and divide it into two parts . . . and take *four* of those parts?" Jeannie asked.

"I'm confused," Betsy said.

"Me too," Jeannie said.

Just then, Sheena walked up. "Four *halves*, isn't it?"

"Yeah!" Betsy exclaimed. "Four *halves*! Halves are two parts.

So . . ."

"So we need two cookies and cut them each in half, then we have four halves," Jeannie said. "One, two, three, four. Twoths. I mean halves."

While Deborah worked on the puzzle of how to be an effective teacher, another question pulsed in the back of her mind: Why hadn't she learned any of this before? As a double major in French and elementary education, she'd taken a methods class, supposedly about how to teach math. Later, of course, she'd taken nearly th⏾tire strand of university-level math classes. But none

of these classes had prepared her to help children learn math. That class did not exist.

The trouble, she suspected, lay in the kind of knowledge one needed to teach well. It fit in neither the category of general education nor that of pure math, though both kinds of knowledge were helpful. In addition to the math itself, she reasoned, math teachers needed to know the kinds of activities and tasks that turned a student's slippery intuition into solid understanding. Not only did they have to master procedures, concepts, and the special cycle of conjecture to argument to proof, but they also had to know the students: how much they were capable of; the iterative, circling way in which they learned; and the kinds of representations—the particular configurations of pictures, numbers, and blocks—that best helped them to understand.

No wonder the class did not exist. It would have had to teach a subject with no name. Even Deborah—who was now both a teacher at Spartan Village and the special "math helping teacher" for all East Lansing elementary schools, not to mention a doctoral student at Michigan State's College of Education—could not articulate the parameters of this knowledge. But that began to change in the mid-1980s, when she decided to study teachers' mathematical knowledge as part of her dissertation.

Her hunch was that Michigan State was still not equipping future teachers with the knowledge and techniques they would need. But to be sure, she devised a test, a short set of teaching problems that she thought math teachers should be able to answer, and gave it to education majors about to graduate.

One question described a group of eighth-grade teachers who "noticed that several of their students were making the same mistake." When multiplying large numbers, like 123 × 645, their students "seemed to be forgetting to 'move the numbers.'" Their work looked like this: